AN ANALYSIS OF A PRECONDITIONER FOR THE DISCRETIZED PRESSURE EQUATION ARISING IN RESERVOIR SIMULATION

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Abstract. We analyze the use of fast solvers as preconditioners for the discretized pressure equation arising in reservoir simulation. Under proper conditions on the permeability functions and the source term, we show that the number of iterations for the Conjugate Gradient method is bounded independently of both the lower bound δ of the permeability and the discretization parameter h. Such results are obtained for a special class of self-adjoint second order elliptic problems with discontinuous coefficients. We also discuss how fast solvers can be utilized in the presence of non-rectangular domains by applying a domain imbedding procedure. The theoretical results are illustrated and supplemented by a series of numerical experiments.

Key words. reservoir simulation, second order elliptic equations, fast solvers, the conjugate gradient method, domain imbedding.

AMS subject classifications. 65N22, 65F10, 65F15.

1. Introduction. The purpose of this paper is to analyze the use of fast solvers as preconditioners for discretized elliptic problems arising in reservoir simulation. Consider the following prototypical second order elliptic boundary value problem,

$$\nabla \cdot (K \nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

with Dirichlet boundary conditions. Here K is a given uniformly positive function defined on Ω. For this problem, it is well known that the associated linear system

$$A_h u_h = f_h \tag{1.2}$$

obtained e.g. by a finite element discretization of (1.1), can be effectively preconditioned by the operator L_h arising from the case of $K = 1$. In fact, the spectral condition number of the preconditioned operator is bounded by

$$\kappa = \frac{\sup_{x \in \Omega} K(x)}{\inf_{x \in \Omega} K(x)},$$

and thus is bounded independently of the mesh parameter h. Since the number of Conjugate Gradient iterations is of order $O(\sqrt{\kappa})$, this shows that L_h is a good

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preconditioner provided that the geometry allows for a fast solver of systems of the form

$$L_h v_h = g_h, \tag{1.3}$$

and that $K(x)$ has a small variation. The term fast solver will in this paper refer to any fast solution strategy for the discrete version of the simplified problem (1.3). If the domain is rectangular, we can apply FFT-based direct solvers. This will be further discussed below. But fast solvers may also be defined by a multilevel method applied to (1.3). This allow us to handle more general geometries.

In this paper, we will consider the case of large variations in K and non-rectangular geometries. Such problems arise in many applications, but our main motivation is the pressure equation in reservoir simulation. For that model, K typically varies from $10^{-6} - 10^2$ and the geometry of a reservoir can be quite complicated. However, we will show that the number of CG-iterations can be bounded independently of the jumps in K. More precisely, we obtain such results for problems with domains that can be partitioned into subdomains such that the variation of K is relatively small in each subdomain. Moreover, K is assumed to have a jump discontinuity along the boundaries of these subdomains. Finally, in the case of a homogeneous Neumann boundary condition, we will show how fast solvers can be applied for non-rectangular domains by using a simple domain imbedding procedure.

Preconditioners for the efficient solution of discretized second order elliptic problems have been extensively studied over the last 15 years. The most popular methods can roughly be divided into three classes; Domain Decomposition type methods, Multigrid Methods and Incomplete Factorizations. Reviews and references for these classes can be found in e.g. Chan and Mathew [15] for domain decomposition methods, Bramble [6] or Hackbusch [26] for multigrid methods and Axelsson [2], Bruaset [11] or Chan and Elman [14] for incomplete factorizations.

Also fast solvers are popular preconditioners. The use of such methods in complicated domains requires some sort of domain imbedding procedure. The main idea in the domain imbedding, also referred to as fictitious domain methods, is to imbed irregular domains into larger regular domains on which fast solvers are available. Hence, a fast solver for the problem defined on the regular domain can be used to construct a preconditioner for the original problem. Such techniques have been studied by several authors, see for example Astrakhantsev [1], Buzbee et al. [12], Börgers and Widlund [5], Dryja [20], Marchuk et al. [29] and Glowinski et al. [25].

It is well known that, under proper conditions on the problem in question, precon-

ditioners based on domain decomposition, multigrid or domain imbedding methods may lead to condition numbers of the preconditioned operator independent of the mesh parameter h. Another important issue is to establish bounds on the number of iterations independent of the jumps in the coefficient functions. This problem was studied in [22] by Dryja, Sarkis and Widlund. They showed that if the coarse grid in a multilevel Schwarz method has grid lines along the jump discontinuities in the coefficient K, then it is possible to obtain a bound on the condition number independently of the size of these jumps, see also Dryja [21] and Bramble et al. [9].

In this paper, we will show similar results for conforming finite element methods as long as fast solvers are available for problems of the form (1.3). More precisely, we show that the number of CG-iterations is bounded independently of both the mesh size and the jumps in the coefficient function provided that the source term is zero in certain regions. The convergence result is derived in a norm that does not depend on the lower bounds on the coefficient K. The main observation in order to prove this result is that the CG-iterations stay within a certain subspace spanned by a set of eigenvectors whose associated eigenvalues are uniformly bounded with respect to both the mesh size and the variations of the coefficient K. This property follows from an assumption on the source term f which turns out to be quite realistic in reservoir simulation. We will also show that proper fast solvers for these problems can be defined on complicated domains by a domain imbedding approach.

The outline of the paper is as follows. In the next section we define our model problem along with the necessary assumptions on the physical parameters. Thereafter, the necessary notation used throughout this paper is presented together with the finite element discretization of our model problem. Sections 3 and 4 contain the theoretical part of this paper and Section 5 contains the numerical experiments.

2. A model problem. The pressure equation arising in reservoir simulation can be written in the form

$$\nabla \cdot [\Lambda (\nabla P - \rho g \nabla D)] + \frac{q}{\rho} = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.1}$$

see for instance Ewing [23] or Peaceman [31]. Here, P represents the unknown fluid pressure related to incompressible or steady state flow in a heterogeneous reservoir, the gravitational constant is given by g, ρ is the density of the fluid in question and D is a function representing the depth of the reservoir measured in the direction of gravity. In this paper, we will assume that g and ρ are constant over the domain Ω. The mobility tensor Λ represents physical parameters such as the viscosity of the fluid and the permeability of the reservoir. Finally, the function q in (2.1) models injection

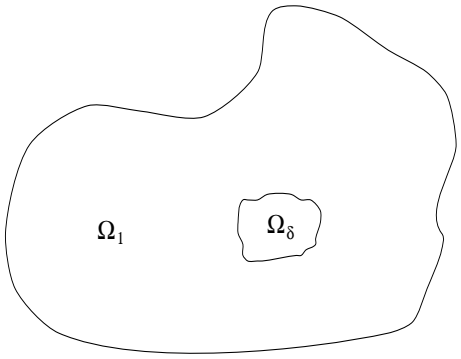

FIG. 2.1. *A prototypical solution domain Ω with subdomains Ω_1 and Ω_δ. Here, Ω_δ represents a low-permeable region with $O(\delta)$ mobility.*

and production wells located inside Ω; i.e. source terms.

Oil reservoirs will normally contain low-permeable zones. In these regions the mobility Λ is close to zero, and the stiffness matrix obtained from a finite element discretization of a problem in the form (2.1) will, in general, be badly conditioned. As mentioned above, this is the case even if the problem is preconditioned with the Laplace operator or some operator spectrally close to the Laplace operator. In this section, we define a model problem, representing a class of second order elliptic problems, for which the preconditioned Conjugate Gradient (CG) method turns out to be an efficient numerical method. The preconditioned operator is badly conditioned, but it turns out that the efficiency of the CG-method, applied to this particular problem, only depends on a bounded subset of the spectrum of the preconditioned operator.

We assume that the domain Ω is the union of two subdomains Ω_1 and Ω_δ with a common boundary $\partial\Omega_\delta$. Here, Ω_δ represents a low-permeable zone with $O(\delta)$ mobility and Ω_1 is a region with $O(1)$ mobility. An example of a solution domain of this type is shown in Figure 2.1. More precisely, $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$ and

$$\Lambda_\delta(x) = \begin{cases} \Lambda(x) & \text{for } x \in \Omega_1 \\ \delta\Lambda(x) & \text{for } x \in \Omega_\delta, \end{cases} \tag{2.2}$$

where $0 < \delta \ll 1$ and Λ is a $O(1)$ mobility tensor defined throughout Ω. A more precise assumption on Λ will be stated below.

By putting $p = P - \rho g D$ we rephrase our model problem in the following form

$$\nabla \cdot (\Lambda_\delta \nabla p) + f = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{2.3}$$

where $f = q/\rho$. We assume that the boundary $\partial\Omega$ is sufficiently smooth and we divide it into two disjoint segments $\Gamma_{\rm d}$ and $\Gamma_{\rm n}$ such that $\partial\Omega = \Gamma_{\rm d} \cup \Gamma_{\rm n}$. The boundary

conditions are then given by

$$p = 0 \quad \text{on } \Gamma_{\mathrm{d}},$$
$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\mathrm{n}}. \tag{2.4}$$

Furthermore, the Darcy velocities are given by

$$\mathbf{v} = -\Lambda_\delta \nabla p. \tag{2.5}$$

Finally we want to state an assumption on the source term $f = q/\rho$. Since q represents either injection or production wells, it is reasonable to assume that $q = 0$ in the area of low permeability. Hence, we find it reasonable to assume that

$$f|_{\Omega_\delta} = 0 \tag{2.6}$$

throughout this paper. This assumption is vital in obtaining sharp bounds in the error analysis for the CG-method.

Remarks.

1. Our results are also valid for non-homogeneous boundary conditions but to obtain a simpler notation, we will only consider the homogeneous case, see (2.4).
2. In the theoretical part of this paper, we assume that there is only one subdomain Ω_δ in the domain Ω where the mobility is of order $O(\delta)$. Of course, in real-world simulations there can be a number of such subdomains. However, the analysis presented here can be extended to the case of a finite number of subdomains Ω_{δ_i} with $O(\delta_i)$ mobility.
3. In the theoretical part of the present paper we consider two dimensional models. However, it should be noted that similar results hold in the case of three space dimensions. Moreover, at the end of this paper we will present a numerical experiment for a three dimensional model problem.
4. Problems of the form (2.3), (2.4) are discussed in [30] where we are concerned with convergence of $\{p_\delta\}$ as $\delta \to 0$, cf. also [10].

2.1. Notation and discretization. In this paper the $L^2(\Omega)$ inner product is denoted by

$$(\psi, \phi) = \int_\Omega \psi\phi \, dx,$$

and the classical Sobolev space $H^1(\Omega)$ is as usual defined as the space of distributions with square-integrable first derivatives. The appropriate subspace for our model

problem, due to the boundary condition (2.4), is

$$V_\Omega = \left\{ \psi \in H^1(\Omega);\ \psi = 0 \text{ on } \Gamma_{\rm d} \right\}.$$

Next, we will assume that the subdomains Ω_1 and Ω_δ have sufficiently smooth boundaries and that $\Omega = \Omega_1 \cup \overline{\Omega}_\delta$, $\Omega_1 \cap \Omega_\delta = \emptyset$ and $\partial\Omega \cap \overline{\Omega}_\delta = \emptyset$. That is, $\overline{\Omega}_\delta$ is contained in Ω and $\mathrm{dist}(\overline{\Omega}_\delta, \partial\Omega) > 0$. Moreover, it should be noted that $\partial\Omega \subset \partial\Omega_1$, $\partial\Omega_\delta \subset \partial\Omega_1$ and $\partial\Omega \cup \partial\Omega_\delta = \partial\Omega_1$.

Finally, we will assume that $f \in L^2(\Omega)$ satisfies (2.6), and that $\Lambda(x) = (\lambda_{i,j}(x))$ is a symmetric uniformly positive definite matrix with entries in $L^\infty(\Omega)$. That is

$$0 < m \le \frac{\mathbf{z}^T \Lambda(x) \mathbf{z}}{|\mathbf{z}|^2} \le M \quad \text{for all } \mathbf{z} \in \mathbb{R}^2 \setminus \{0\} \text{ and } x \in \Omega, \tag{2.7}$$

where m and M are finite constants independent of δ and x. Here, $|\mathbf{z}|$ for $\mathbf{z} \in \mathbb{R}^2$ denotes the Euclidean norm of $\mathbf{z}$.

Clearly, (2.7) implies that our model problem is strictly elliptic for any $\delta > 0$, see for instance Dautray and Lions [17, Ch. II.8]. Hence, if the boundary of Ω is sufficiently smooth, it follows from the Lax-Milgram theorem, see for example Dautray and Lions [16] or Gilbarg and Trudinger [24], that the following weak formulation of (2.3), (2.4) has a unique solution: Find $p \in V_\Omega$ such that

$$\int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla p)\, dx = \int_\Omega f\psi\, dx \quad \text{for all } \psi \in V_\Omega. \tag{2.8}$$

Now we want to define a Ritz-Galerkin discretization of (2.8). To this end, let $\{N_1, \ldots, N_q\}$ be a set of linearly independent functions such that $N_i \in V_\Omega$ for $i = 1, \ldots, q$, and define

$$V_{\Omega,h} = \mathrm{span}\{N_1, \ldots, N_q\}.$$

Here, the subscript $h \in I$, where I is some subset of $\mathbb{R}_+$, is used to distinguish the finite dimensional entities from the corresponding symbols used in the continuous case. Typically, h is the mesh size for a grid defined on Ω. With this notation at hand, we define the approximation of (2.8) as follows: Find $p_h \in V_{\Omega,h}$ such that

$$a_\delta(p_h, \psi) = (\psi, f) \quad \text{for all } \psi \in V_{\Omega,h}, \tag{2.9}$$

where $a_\delta(\cdot,\cdot)$ is the bilinear form defined on $V_\Omega \times V_\Omega$ by

$$a_\delta(\varphi, \psi) = \int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla\varphi)\, dx.$$

We need two specific assumptions on $V_{\Omega,h}$. In order to motivate the first assumption for the discrete problems, we start by considering a similar feature in the

continuous case. To this end, consider the following boundary value problem defined on Ω_δ: Find $u \in H^1(\Omega_\delta)$ such that $u = w \in H^{1/2}(\partial\Omega_\delta)$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla v \cdot (\Lambda \nabla u)\, dx = 0 \quad \text{for all } v \in H_0^1(\Omega_\delta).$$

Under proper conditions on Ω_δ, this problem has an unique solution u that satisfies

$$\| u \|_{H^1(\Omega_\delta)} \leq c_0 \| w \|_{H^{1/2}(\partial\Omega_\delta)},$$

where c_0 is a constant independent of δ, see e.g. Hackbusch [27, Ch. 7.3]. Motivated by this property, valid in the continuous case, we assume that a similar property holds in the discrete case. To state the assumption more precisely, we introduce the spaces

$$V_{\Omega_\delta,h} = \text{span}\{N_1|_{\Omega_\delta}, \ldots, N_q|_{\Omega_\delta}\},$$
$$V_{\Omega_1,h} = \text{span}\{N_1|_{\Omega_1}, \ldots, N_q|_{\Omega_1}\},$$

and let $T_{\Omega_1} : H^1(\Omega_1) \to H^{1/2}(\partial\Omega_1)$ denote the trace operator. Since $\partial\Omega_\delta \subset \partial\Omega_1$ we can introduce the set

$$G_{\Omega_1,h} = \{T_{\Omega_1}(\psi)|_{\partial\Omega_\delta};\ \psi \in V_{\Omega_1,h}\}.$$

Then the first assumption is:

A. For every $h \in I$ and every $w \in G_{\Omega_1,h}$ the following problem has a unique solution:

Find $u_h \in V_{\Omega_\delta,h}$ such that $u_h = w$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla u_h)\, dx = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta). \tag{2.10}$$

Furthermore, there exists a constant c_1 independent of δ and h such that

$$\| u_h \|_{H^1(\Omega_\delta)} \leq c_1 \| w \|_{H^{1/2}(\partial\Omega_\delta)}.$$

Under proper conditions on Ω_δ, assumption **A** can be established for various types of finite element spaces, see Bramble, Pasciak and Schatz [7] and [8].

Next, we will assume that;

B. If $\varphi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta)$ then the function

$$\psi = \begin{cases} \varphi & \text{on } \Omega_\delta \\ 0 & \text{on } \Omega_1 \end{cases}$$

belongs to $V_{\Omega,h}$.

This condition makes it possible to extend discrete test functions defined on Ω_δ, and that vanish on $\partial\Omega_\delta$, to the entire domain Ω.

Typically, these assumptions are likely to be satisfied if the interface $\partial\Omega_\delta$ coincides with grid-lines of the mesh associated with the finite element space $V_{\Omega,h}$.

2.2. Preconditioning. We will now introduce a preconditioner for the discrete problem (2.9). Roughly speaking, the preconditioner is defined by introducing an auxiliary problem of the same type but with $\delta = 1$. We will show that this preconditioner leads to uniform bounds on the number of iterations for the CG-method provided that condition (2.6) is satisfied. Furthermore, we will show that if (2.6) is violated then the number of iterations is bounded by as $O(\ln \delta^{-1})$. Practical applications of this preconditioner are discussed in Section 4 and Section 5.

Let $f_h \in V_{\Omega,h}$ denote the L^2-projection of f in $V_{\Omega,h}$. Since $(f_h, \psi) = (f, \psi)$ for all $\psi \in V_{\Omega,h}$ it follows that the problem (2.9) is equivalent to the following problem: Find $p_h \in V_{\Omega,h}$ such that

$$A_{\delta,h} p_h = f_h. \tag{2.11}$$

Here $A_{\delta,h} : V_{\Omega,h} \to V_{\Omega,h}$ is the linear operator associated with the bilinear form $a_\delta(\cdot,\cdot)$ defined by

$$(A_{\delta,h}\varphi, \psi) = a_\delta(\varphi, \psi) \quad \text{for all } \varphi, \psi \in V_{\Omega,h}. \tag{2.12}$$

Now we define a preconditioner for the linear system (2.11) by introducing the linear operator $M_h : V_{\Omega,h} \to V_{\Omega,h}$ determined by $a_1(\cdot,\cdot)$, i.e.

$$(M_h\varphi, \psi) = \int_\Omega \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx \quad \text{for all } \varphi, \psi \in V_{\Omega,h}. \tag{2.13}$$

Preconditioning with M_h, we get the system

$$M_h^{-1} A_{\delta,h} p_h = M_h^{-1} f_h. \tag{2.14}$$

We will discuss below how this preconditioning can be implemented in various cases, cf. Section 4.

3. Theoretical results. In this section we will study the efficiency of the CG-method applied to the problem (2.14). The main idea of the analysis is to utilize a certain invariance property of the operator $M_h^{-1} A_{\delta,h}$: There is a subspace of $V_{\Omega,h}$, containing the solution p_h of (2.11), such that if the start vector $p^{(0)}$ is in the subspace, then all subsequent CG-approximations $p^{(k)}$ also belong to this subspace. By following the standard argument for the error-bound of the CG-method, we utilize this invariance property to show that the effective condition number of the operator is defined by the set of eigenvalues associated with the eigenvectors spanning the invariant subspace. Since these eigenvalues can be bounded within a fixed interval $[\lambda_0, 1]$ independent of δ and h, we get a uniform bound on the number of CG-iterations.

Furthermore, we will show that the eigenvalues of $M_h^{-1}A_{\delta,h}$ satisfy $\lambda \in \{\delta\} \cup [\lambda_0, 1]$, where λ_0 is a positive constant independent of δ and h. This observation is used to bound the number of CG-iterations also in cases where the assumption (2.6) is violated. In this latter case, the bound is weaker and allows a $O(\ln \delta^{-1})$ growth in the number of iterations.

3.1. Properties of the preconditioner. Consider the linear operator $M_h^{-1}A_{\delta,h} : V_{\Omega,h} \to V_{\Omega,h}$ and the inner product

$$[\varphi, \psi] = (M_h\varphi, \psi) \quad \text{for all } \varphi, \psi \in V_{\Omega,h}. \tag{3.1}$$

Let S_δ be the set of functions with support in $\overline{\Omega}_\delta$ and let S_1 be its orthogonal complement with respect to the inner product $[\cdot,\cdot]$, i.e.

$$S_\delta = \{\psi \in V_{\Omega,h};\ \mathrm{supp}(\psi) \subset \overline{\Omega}_\delta\} \text{ and } S_1 = S_\delta^\perp.$$

Since S_δ is closed, it follows that $V_{\Omega,h}$ is the direct sum of S_δ and S_1, $V_{\Omega,h} = S_\delta \oplus S_1$.

The next two lemmas state the properties of the operator $M_h^{-1}A_{\delta,h}$. Recall that $\Lambda_\delta(x) = \delta\Lambda(x)$ for all $x \in \Omega_\delta$, see (2.2). This property of Λ_δ leads to the following result:

LEMMA 3.1. *Let the linear operators $A_{\delta,h}$ and M_h be defined in (2.12) and (2.13), respectively. Then*

a) $M_h^{-1}A_{\delta,h}\varphi = \delta\varphi$ *for all* $\varphi \in S_\delta$,

b) $M_h^{-1}A_{\delta,h}\varphi \in S_1$, *for all* $\varphi \in S_1$.

Proof.

Part a). Let $\varphi \in S_\delta$ be arbitrary. For any $\psi \in V_{\Omega,h}$ it follows from the assumption that $\mathrm{supp}(\varphi) \subset \overline{\Omega}_\delta$ and the property (2.2) of Λ_δ that

$$\begin{aligned}(A_{\delta,h}\varphi, \psi) = \int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla\varphi)\, dx = \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx = \\ \delta \int_\Omega \nabla\psi \cdot (\Lambda\nabla\varphi)\, dx = [\delta\varphi, \psi] = (M_h\delta\varphi, \psi).\end{aligned} \tag{3.2}$$

Hence, since (3.2) holds for all $\psi \in V_{\Omega,h}$ we conclude that

$$A_{\delta,h}\varphi = M_h\delta\varphi,$$

which proves part a).

Part b). From the definition (2.12) of $A_{\delta,h}$ and the properties (2.2) and (2.7) of Λ_δ it follows that

$$(A_{\delta,h}z, z) = 0 \Rightarrow z = 0.$$

Now, let $\varphi \in S_1$ be arbitrary and assume that $\Phi = M_h^{-1} A_{\delta,h}\varphi \in S_\delta$. Consequently,

$$M_h \Phi = A_{\delta,h}\varphi,$$

and since $S_1 = S_\delta^\perp$ with respect to the $(M_h \cdot, \cdot)$ inner product we find that

$$(A_{\delta,h}\varphi, \varphi) = (M_h \Phi, \varphi) = 0,$$

which in turn implies that $\varphi = 0$. Thus, the only element of S_1 that is mapped by $M_h^{-1} A_{\delta,h}$ into S_δ is the 0-function. Hence, we conclude that $M_h^{-1} A_{\delta,h}\varphi \in S_1$ for all $\varphi \in S_1$. □

We observed above that δ is a multiple eigenvalue of $M_h^{-1} A_{\delta,h}$. In fact the multiplicity is given by $\alpha = \dim(S_\delta)$. Next we show that the remaining eigenvalues can be bounded in an interval $[\lambda_0, 1]$, where $\lambda_0 > 0$ is independent of δ and h.

LEMMA 3.2. *If $v \in S_1$ is an eigenfunction of $M_h^{-1} A_{\delta,h}$ with eigenvalue λ then*

$$\lambda_0 \leq \lambda \leq 1,$$

where $\lambda_0 = m(M + Mc_2)^{-1}$ (see (2.7) and (3.4)), i.e. λ_0 is a positive constant independent of δ and h. Here, $A_{\delta,h}$ and M_h are the operators defined in (2.12) and (2.13), respectively.

Proof. Since $v \in S_1 = S_\delta^\perp$, it follows that $v \perp \psi$ for every $\psi \in S_\delta$. That is

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla v)\, dx = \int_{\Omega} \nabla\psi \cdot (\Lambda \nabla v)\, dx = 0 \quad \text{for all } \psi \in S_\delta. \tag{3.3}$$

Let $w = T_{\Omega_1}(v)|_{\partial\Omega_\delta}$ be the boundary values of v at $\partial\Omega_\delta$. Then it follows from assumption **B** and equation (3.3) that $\tilde{v} = v|_{\Omega_\delta}$ solves the problem; Find $\tilde{v} \in V_{\Omega_\delta,h}$ such that $\tilde{v} = w$ on $\partial\Omega_\delta$ and

$$\int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla \tilde{v})\, dx = 0 \quad \text{for all } \psi \in V_{\Omega_\delta,h} \cap H_0^1(\Omega_\delta).$$

Thus, according to assumption **A** we have

$$\| v \|_{H^1(\Omega_\delta)} \leq c_1 \| T_{\Omega_1}(v) \|_{H^{1/2}(\partial\Omega_\delta)},$$

and consequently, by a trace inequality, we have

$$\| v \|_{H^1(\Omega_\delta)} \leq c_1 \, \| T_{\Omega_1} \| \, \| v \|_{H^1(\Omega_1)}.$$

Here $\| T_{\Omega_1} \|$ is bounded independently of δ and h, cf. e.g. Hackbusch [27], and we can apply Poincaré's inequality to get

$$\int_{\Omega_\delta} |\nabla v|^2\, dx \leq c_2 \int_{\Omega_1} |\nabla v|^2\, dx, \tag{3.4}$$

where c_2 is constant independent of δ and h.

Next, $M_h^{-1}A_{\delta,h}v = \lambda v$ implies that

$$\lambda \int_\Omega \nabla v \cdot (\Lambda \nabla v)\, dx = \int_\Omega \nabla v \cdot (\Lambda_\delta \nabla v)\, dx \tag{3.5}$$

Therefore, from (2.2), (2.7) and (3.4) we get

$$\begin{aligned}\lambda &\geq \frac{\int_{\Omega_1} \nabla v \cdot (\Lambda_\delta \nabla v)\, dx}{M \int_{\Omega_1} |\nabla v|^2\, dx + M \int_{\Omega_\delta} |\nabla v|^2\, dx} \geq \frac{m \int_{\Omega_1} |\nabla v|^2\, dx}{M \int_{\Omega_1} |\nabla v|^2\, dx + Mc_2 \int_{\Omega_1} |\nabla v|^2\, dx} \\ &= \frac{m}{M + Mc_2} = \lambda_0,\end{aligned}$$

which proves the left inequality in the lemma.

The right inequality follows from (3.5), the property (2.2) of Λ_δ and the assumption that $\delta \leq 1$. □

The following Corollary summarizes Lemma 3.1 a) and Lemma 3.2.

COROLLARY 3.3. *The eigenvalues of $M_h^{-1}A_{\delta,h}$ are contained in $\{\delta\} \cup [\lambda_0, 1]$. Here λ_0 is a positive constant independent of δ and h, and the eigenvalue δ has multiplicity $\alpha = \dim(S_\delta)$.*

3.2. Convergence of the preconditioned Conjugate Gradient method. Now we are ready to analyze the efficiency of the CG-method applied to (2.14). The preconditioning is incorporated in the CG-algorithm by applying the inner product $[\cdot, \cdot]$ defined in (3.1). Clearly, by Lemma 3.1 part b), S_1 is an invariant subspace for $M_h^{-1}A_{\delta,h}$. This implies that the well-known Krylov subspaces generated in the CG-method are subspaces of S_1, provided that the initial guess $p^{(0)}$ in the iteration process satisfies $p^{(0)} \in S_1$.

Furthermore, from (2.2) and assumption (2.6) we find that the solution p_h of (2.9) satisfies

$$\delta \int_{\Omega_\delta} \nabla \psi \cdot (\Lambda \nabla p_h)\, dx = 0 \quad \text{for all } \psi \in S_\delta.$$

That is, $[\psi, p_h] = 0$ for all $\psi \in S_\delta$ and hence

$$p_h \in S_1. \tag{3.6}$$

Having established (3.6) and the invariance property of S_1, it follows in general, cf. e.g. Bramble [6] or Stoer and Bulirsch [33], that only the eigenvalues associated the eigenvectors spanning S_1 have influence on the number of CG-iterations needed to solve (2.14). Hence, by applying Lemma 3.2 and following the steps of the standard

argument for the error-bound of the CG-method we get an error bound for the CG-method stated in the usual norm

$$\|\psi\|_{A_\delta} = \sqrt{(A_{\delta,h}\psi, \psi)} \quad \text{for } \psi \in V_{\Omega,h}. \tag{3.7}$$

THEOREM 3.4. *Let $p^{(k)}$ be the k'th approximation to the solution p_h of (2.9) in the CG-method applied to (2.14). If $p^{(0)} \in S_1$ then*

$$\|p_h - p^{(k)}\|_{A_\delta} \leq 2 \left(\frac{1 - \sqrt{\lambda_0}}{1 + \sqrt{\lambda_0}} \right)^k \|p_h - p^{(0)}\|_{A_\delta},$$

where λ_0 is a constant independent of δ and h.

From this result it follows immediately that the number of CG-iterations needed to achieve a given accuracy, measured in the $\|\cdot\|_{A_\delta}$ norm, is bounded by $O(\lambda_0^{-1/2})$, cf. e.g. [2]. Moreover, $0 \in S_1$ and hence it is never a problem to find an initial guess $p^{(0)}$ in S_1.

As mentioned above, the proof of this theorem is analogous to the standard error analysis of the CG-method and therefore omitted, see e.g. [6, 33].

A remark on the norm. Clearly the norm $\|\cdot\|_{A_\delta}$ defined on $V_{\Omega,h}$ depends on δ, cf. (3.7). In particular, as $\delta \to 0$, $A_{\delta,h}$ becomes singular and consequently the norm $\|\cdot\|_{A_\delta}$ is very weak for small values of δ. Thus one might fear that small errors in $\|\cdot\|_{A_\delta}$ do not necessarily imply a good approximation. This is, however, not the case for the approximations $p^{(k)}$ generated by the CG-method. It follows by the invariance property of S_1, established above, that $p^{(k)} - p^{(0)} \in S_1$ for all k. On this subspace the norms $\|\cdot\|_{A_\delta}$ and $\|\cdot\|_{A_1}$, defined by putting $\delta = 1$ in (3.7), are equivalent. Hence, we obtain the following result.

COROLLARY 3.5. *If the CG-method is applied to (2.14) then the k'th approximation $p^{(k)}$ to the solution p_h of (2.9) satisfies*

$$\|p_h - p^{(k)}\|_{A_1} \leq 2\, C \left(\frac{1 - \sqrt{\lambda_0}}{1 + \sqrt{\lambda_0}} \right)^k \|p_h - p^{(0)}\|_{A_1},$$

provided that $p^{(0)} \in S_1$. Here, C and λ_0 are constants independent of δ and h.

This result shows that the error is bounded in a norm which does not depend on δ, and is hence stronger than the error estimate obtained in Theorem 3.4.

Proof. Recall that $0 < \delta \ll 1$, and thus

$$\begin{aligned} \|\psi\|_{A_\delta}^2 &= \int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla\psi)\, dx = \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla\psi)\, dx + \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla\psi)\, dx \\ &\leq \int_{\Omega_1} \nabla\psi \cdot (\Lambda \nabla\psi)\, dx + \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda \nabla\psi)\, dx = \|\psi\|_{A_1}^2 \end{aligned} \tag{3.8}$$

for all $\psi \in V_{\Omega,h}$.

Let $\psi \in S_1$ be arbitrary. Then it follows, as in the proof of Lemma 3.2, that there exists a constant c_2, independent of δ and h, such that

$$\int_{\Omega_\delta} |\nabla\psi|^2 \, dx \le c_2 \int_{\Omega_1} |\nabla\psi|^2 \, dx, \tag{3.9}$$

see equations (3.3)-(3.4). Now, (2.7) and (3.9) imply that

$$\begin{aligned}
\|\psi\|_{A_1}^2 &= \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx + \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx \\
&\le \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx + c_2 M \int_{\Omega_1} \nabla\psi \cdot \nabla\psi \, dx \\
&\le \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx + \frac{c_2 M}{m} \int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx \\
&\le \left(1 + \frac{c_2 M}{m}\right) \left(\int_{\Omega_1} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx + \delta \int_{\Omega_\delta} \nabla\psi \cdot (\Lambda\nabla\psi) \, dx \right) \\
&= \left(1 + \frac{c_2 M}{m}\right) \|\psi\|_{A_\delta}^2 \quad \text{for all } \psi \in S_1. \qquad (3.10)
\end{aligned}$$

Hence, from (3.8) and (3.10) we conclude that there exists a constant C, independent of h and δ, such that

$$C \, \|\psi\|_{A_1} \le \|\psi\|_{A_\delta} \le \|\psi\|_{A_1} \quad \text{for all } \psi \in S_1. \tag{3.11}$$

Recall that if $p^{(0)} \in S_1$ then the invariance property of S_1 implies that $p^{(k)} \in S_1$ for all k. Furthermore, from (3.6) we conclude that $p_h - p^{(k)} \in S_1$ for all k. Thus, the desired result follows from (3.11) and Theorem 3.4. □

3.3. General source terms; a convergence result. A crucial condition for Theorem 3.4 is assumption (2.6) on the source term f. In this section we will briefly discuss general source terms, i.e. source terms f that only satisfy $f \in L^2(\Omega)$. Clearly, Corollary 3.3 is still valid in this case. Hence, the eigenvalues of $M_h^{-1} A_{\delta,h}$ satisfy

$$\lambda \in \{\delta\} \cup [\lambda_0, 1], \tag{3.12}$$

where we recall that λ_0 is a positive constant independent of δ and h.

Positive definite problems with eigenvalue distributions of the form (3.12) have been thoroughly studied by Axelsson and Lindskog [3, 4]. They showed that an error bound of the form

$$\frac{\|p_h - p^{(k)}\|_{A_\delta}}{\|p_h - p^{(0)}\|_{A_\delta}} \le \xi$$

will be reached in at most

$$k = \frac{\ln(2/\xi) + \ln(1/\delta)}{\ln(\sigma^{-1})} + 1 \tag{3.13}$$

CG-iterations. Here, $\xi > 0$ is the error level and σ is given by

$$\sigma = \frac{1 - \sqrt{\lambda_0}}{1 + \sqrt{\lambda_0}}.$$

Thus we observe that the number of CG-iterations increases as $O(\ln(1/\delta))$ in this case, and that the norm here depends on δ. So we conclude that our results are much sharper in the cases where (2.6) is satisfied.

4. The preconditioning scheme. We have observed above that linear systems of the form

$$A_{\delta,h} p_h = f_h \tag{4.1}$$

where $A_{\delta,h}$ is defined by (2.12), can be effectively preconditioned by the operator M_h defined in (2.13). In order to apply this preconditioner, we have to be able to solve systems of the form

$$M_h v_h = g_h \tag{4.2}$$

efficiently, since this operation is needed within each CG-iteration. For some problems this is straightforward; consider e.g. the case of $\Lambda = I$, where I is the identity matrix and $\Omega = R$, where R is a rectangle. Then (4.2) reduces to

$$L_h v_h = g_h \tag{4.3}$$

where L_h is the discrete Laplacian. Problems of the form (4.3) posed on a rectangle can be very effectively solved using FFT-based fast solvers, c.f. Buzbee et al. [13], Dorr [18] or Swarztrauber [32]. Thus in the case of $\Lambda = I$ and $\Omega = R$, we have a good preconditioner. It should be noted here that, although we limit our discussion to FFT-based fast solvers on rectangular domains, this preconditioning strategy is applicable whenever a fast solver is available. Hence, by using multigrid methods or domain decomposition based methods, we can handle more general domains. Furthermore, we will show below that for homogeneous Neumann type boundary conditions very general domains can be handled using domain imbedding.

Next we consider the case of a general tensor Λ satisfying the requirement (2.7), but we still assume that $\Omega = R$. Problems of the form (4.2), posed on a rectangle R, can be preconditioned by the operator L_h in (4.3). More precisely, within each CG-iteration for solving (4.1), systems of the form (4.2) have to be solved. This system can be solved using the CG-method, defining an inner iteration, with L_h as a preconditioner. We have already stated that systems of the form (4.3) can be solved very fast. Furthermore, the spectral condition number of $L_h^{-1} M_h$ satisfies

$$\kappa(L_h^{-1} M_h) \leq M/m,$$

where M and m are defined in (2.7). Thus the inner iteration converges in $O(\sqrt{M/m})$ iterations independently of h and δ.

From these considerations, it follows that we have a good preconditioner for (4.1) whenever Ω is a rectangle. Next we discuss how this technique can be applied in the case of non-rectangular domains in reservoir simulation.

4.1. Domain imbedding. In order to apply this preconditioning strategy to the pressure equation arising in reservoir simulation, we have to reconsider the boundary condition (2.4). We assume now that the reservoir is surrounded by non-permeable rock. Thus $\Gamma_{\rm d} = \emptyset$ and $\Gamma_{\rm n} = \partial\Omega$, i.e. we have a no-flow boundary condition on the entire boundary. This assumption on the boundary condition facilitates the application of a domain imbedding procedure to handle complicated geometries.

The domain imbedding procedure is defined by introducing a small regularization parameter ϵ defining the mobility in $R - \Omega$ where R is a rectangle covering Ω. Thus the mobility tensor is extended to the new domain by putting it equal to ϵ times the identity outside the original domain, see Figure 4.1. As ϵ tends to zero, the fluid flow from Ω into $\Omega_\epsilon = R - \Omega$ tends towards zero. That is, the pressure and the velocity of the problem defined on R converge towards the solution and velocity of the original problem as ϵ tends to zero, see [30]. Moreover, the theory presented above will lead to convergence results for for the CG-method independent of the regularization parameter ϵ.

Let us give a slightly more detailed presentation of the domain imbedding procedure. We will study equation (2.3) with a homogeneous Neumann boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{\rm n} = \partial\Omega \tag{4.4}$$

in this section. In order to obtain a properly posed problem, the space V_Ω introduced in Section 2 is redefined by

$$V_\Omega = \left\{ \psi \in H^1(\Omega);\ \int_\Omega \psi \, dx = 0 \right\}, \tag{4.5}$$

and the source term f is assumed to satisfy (2.6) along with

$$\int_\Omega f \, dx = 0. \tag{4.6}$$

It is well known that, under these assumptions, the weak formulation of the problem (2.3) and (4.4) has a unique solution $p \in V_\Omega$.

The Ritz-Galerkin discretization of this problem, in terms of a finite dimensional subspace $V_{\Omega,h}$ of V_Ω, is defined in the usual way: Find $p_h \in V_{\Omega,h}$ such that

$$\int_\Omega \nabla\psi \cdot (\Lambda_\delta \nabla p_h) \, dx = \int_\Omega f\psi \, dx \quad \text{for all } \psi \in V_{\Omega,h}. \tag{4.7}$$

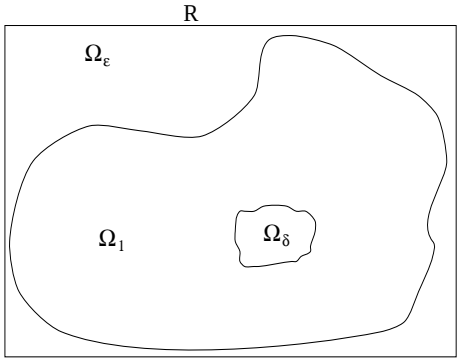

FIG. 4.1. *An example of a domain* $\Omega = \Omega_1 \cup \Omega_\delta$ *imbedded in a rectangular domain* $R = \Omega_\epsilon \cup \Omega_1 \cup \Omega_\delta$.

Now we want to approximate the solution p_h of (4.7) by the solution $p_{\epsilon,h}$ of a problem posed on a rectangular domain R. Since we assume that Ω is bounded, it can be imbedded in a rectangular domain R, i.e. $\Omega \subset R$. Next, we introduce a regularization parameter $\epsilon > 0$ and define an extended mobility tensor $\Lambda_{\epsilon,\delta}$ by

$$\Lambda_{\epsilon,\delta}(x) = \begin{cases} \Lambda_\delta(x) & \text{for } x \in \Omega \\ \epsilon\, I & \text{for } x \in \Omega_\epsilon, \end{cases} \tag{4.8}$$

where I is the identity matrix, and $\Omega_\epsilon = R - \overline{\Omega}$, see Figure 4.1. Let $\overline{f}$ denote the canonical extension of f to R by putting $\overline{f} = 0$ on Ω_ϵ. Then the approximation $p_{\epsilon,h}$ to p_h is defined as follows: Find $p_{\epsilon,h} \in V_{R,h}$ such that

$$\int_R \nabla\psi \cdot (\Lambda_{\epsilon,\delta} \nabla p_{\epsilon,h})\, dx = \int_R \overline{f}\psi\, dx \quad \text{for all } \psi \in V_{R,h}. \tag{4.9}$$

Here, $V_{R,h}$ is a finite dimensional subspace of

$$V_R = \left\{ \psi \in H^1(R);\ \int_R \psi\, dx = 0 \right\}.$$

Clearly, we are concerned about how well $p_{\epsilon,h}$ approximates p_h on Ω. We studied this approximation problem in a slightly different setting in [30]. By modifying the argument given there, it can be shown that

$$\| p_{\epsilon,h} - p_h \|_{H^1(\Omega)} \leq c\epsilon,$$
$$\| \mathbf{v}_{\epsilon,h} - \mathbf{v}_h \|_{(L^2(\Omega))^2} \leq c\epsilon,$$

where c is a constant independent of ϵ, δ and h. Consequently, we can solve (4.9) instead of (4.7) and only introduce an error of order $O(\epsilon)$. This shows how we can exploit the existence of fast solvers on rectangles also in complicated domains.

Now the preconditioner is defined as follows. Let $A_{\epsilon,\delta,h} : V_{R,h} \to V_{R,h}$ be the linear operator associated with the problem (4.9) and define $M_h = A_{1,1,h}$. Clearly,

solving (4.9) is equivalent to finding $p_{\epsilon,h}$ such that

$$M_h^{-1} A_{\epsilon,\delta,h} p_{\epsilon,h} = M_h^{-1} \overline{f}_h, \tag{4.10}$$

where $\overline{f}_h$ denotes the L^2-projection of $\overline{f}$ in $V_{R,h}$. Then, by a straightforward generalization of the theory presented in Section 3, it follows that the problem (4.10) can be efficiently solved by the CG-method. More precisely, the number of CG-iterations needed to obtain a relative error less than a fixed error level, is independent of ϵ, δ and h, see Theorem 3.4.

We have described how complicated domains can be imbedded in rectangular domains without introducing significant errors. This enables us to apply FFT-based fast direct solvers as preconditioners. But it should also be noted that a similar approach can be used to imbed a complicated domain into a simpler, but not necessarily rectangular domain, and thus facilitating the application of a multilevel method.

5. A computational study of the method. The purpose of this section is to present some examples illustrating the behavior of the preconditioning method described in the previous sections. We will consider both some cases covered by the theory developed above and some examples where we violate some of the assumptions.

In all our experiments we consider a diagonal permeability tensor of the form

$$\Lambda_\delta \;=\; K_\delta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where K_δ is a scalar function. If not stated otherwise, the source term f is given by

$$f(x,y) = \begin{cases} 1 & \text{for } (x,y) \in [0.375, 0.468] \times [0.562, 0.656], \\ -1 & \text{for } (x,y) \in [2.625, 2.718] \times [2.437, 2.531], \\ 0 & \text{elsewhere.} \end{cases}$$

We will only consider problems with a homogeneous Neumann boundary condition, i.e. boundary value problems of the form (2.3), (4.4). This boundary value problem is discretized using standard piecewise linear finite elements. The linear systems (4.1) arising from this discretization are solved as described in Section 4 above. All the implementation is done within the Diffpack framework [19, 28].

In the preconditioning technique described in Section 4, linear systems of the form (4.3) have to be solved very efficiently. This is achieved by using a "fast solver" of the kind discussed by e.g. Buzbee et al. [13], Dorr [18] and Swarztrauber [32]. Although these solvers require $O(n \log(n))$ flops, where $n \sim 1/h^2$, our numerical experiments show that they are significantly faster than e.g. a multigrid method for relevant mesh sizes.

The computations are carried out in double precision. In all our experiments, $p^{(0)} = 0 \in S_1$ is used as initial guess in the iteration process, cf. Theorem 3.4, and the CG-method is halted when

$$\frac{\|f_h - A_{\epsilon,\delta,h} p^{(k)}\|_2}{\|f_h\|_2} \leq 10^{-6}, \tag{5.1}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

5.1. Large variations in smooth coefficients. In the theory developed above, we allow K_δ to have large and discontinuous variations. In fact, we assume that K_δ attains values on certain levels, say $O(1)$ and $O(\delta)$. Before we address such cases computationally, let us consider a smooth function of large variation,

$$K_\delta(x,y) \;=\; \frac{\delta + (x-3/2)^2 (y-3/2)^2}{\delta + (3/2)^4}$$

on the domain $\Omega = [0,3]^2$. Clearly

$$\frac{\sup_{(x,y)\in\Omega} K_\delta(x,y)}{\inf_{(x,y)\in\Omega} K_\delta(x,y)} = \frac{\delta + (3/2)^4}{\delta},$$

and then, by the argument discussed in Section 1, we expect the number of CG-iterations to grow as $O(\delta^{-1/2})$. The number of iterations are given in Table 5.1 and we note that the estimate $O(\delta^{-1/2})$ is fairly accurate.

δ	1/2	1/4	1/8	1/16	1/32
CG-iterations	26	38	55	79	114
Rate	-	-0.54	-0.53	-0.52	-0.53

TABLE 5.1

The relation between the numbers of CG-iterations and δ for a problem with large variations in smooth coefficients. The number of unknowns is $n = 1025 \times 1025$.

Certainly, this shows that the classical bound based on the condition number of the preconditioned operator is sharp in the case of a smooth K_δ. Consequently, for δ sufficiently small, the preconditioner is not very good for such problems. At present, we are unaware of preconditioning techniques that can solve such problems with a number of iterations independent of n and δ.

5.2. A discontinuous coefficient. Next we consider a problem of the type described in Section 2. Define

$$K_\delta(x,y) = \begin{cases} \delta & (x,y) \in \Omega_\delta, \\ 1 & (x,y) \in \Omega - \Omega_\delta, \end{cases}$$

where Ω and Ω_δ are depicted in Figure 5.1.