PORTFOLIO MANAGEMENT AND CORRELATION

FRED ESPEN BENTH, JON GJERDE, AND SIGURD SANNAN

1. INTRODUCTION

The uncertainty of future returns from a portfolio of financial assets is influenced by the correlations among the assets. This paper demonstrates the quantitative effect of market correlations on portfolio management. We look at the optimal allocation and risk in a portfolio as a function of correlations between the assets. Merton's problem is chosen as a toy model to study this, where assets are modelled as multi-dimensional lognormal processes.

In the paper we consider real and synthetic examples of situations where an investor can place money in two correlated assets. We demonstrate the functional dependence of optimal allocation on correlation and compare models taking correlation into account with those who leave correlation aside. As a measure of portfolio risk, we use the concept of Value-at-Risk (VaR) from JP Morgan's RiskMetrics [1]. The importance of taking correlations into account in risk managemenent is clearly demonstrated in our examples. Estimates for VaR on the optimal portfolio is calculated for investors with different risk profiles. Finally, as an application, we calculate the optimal portfolio allocation for a pair of assets traded on Oslo Stock Exchange and demonstrate the effect of introducing correlations in the model. In the last section we discuss some extensions towards more realistic models and give references to relevant papers.

2. MATHEMATICAL MODEL

Let us consider two assets X_t^1 and X_t^2, with $X_0^1 = x_1$ and $X_0^2 = x_2$. Let $\pi_t \in [0,1]$ be the fraction of the total wealth invested in asset X_t^1, and $1 - \pi_t$ in asset X_t^2 and denote by W_t^π the total wealth for a given strategy π. We assume that no short-selling of stocks is allowed in our market. The goal is to find an optimal portfolio allocation strategy π_t^* maximizing the expected utility at time T, i.e.

$$E^{t,w}[U(W_T^{\pi^*})] = \sup_{\pi_t} E^{t,w}[U(W_T^u)],$$

where $U(x) = \frac{x^\gamma}{\gamma}, 0 < \gamma < 1$ is a HARA-utility function[1], and $t < T$. This is known as the Merton problem. We will study this problem when the two assets are geometrical

---

[1]γ, or rather $1 - \gamma$, is the investor's aversion towards risk.

Brownian motions on the form

$$dX_t^1 = X_t^1(\mu_1 \, dt + \sigma_1 \, dB_t^1)$$
$$dX_t^2 = X_t^2(\mu_2 \, dt + \rho\sigma_2 \, dB_t^1 + \sqrt{(1 - \rho^2)}\sigma_2 \, dB_t^2),$$

where B_t^1 and B_t^2 are independent Brownian motions. We have the statistical properties

$$E[X_t^i] = e^{\mu_i t} \; ; \; i \in \{1, 2\}$$
$$\text{Std}[\log X_t^i] = \sigma_i \sqrt{t} \; ; \; i \in \{1, 2\}$$
$$\text{Corr}[\log X_t^1, \log X_t^2] = \rho$$

The wealth process has the dynamics

$$dW_t^\pi = W_t^\pi(\mu_2 + \pi_t(\mu_1 - \mu_2)) \, dt$$
$$+ W_t^\pi(\rho\sigma_2 + \pi_t(\sigma_1 - \rho\sigma_2)) \, dB_t^1 + W_t^\pi(1 - \pi_t)\sqrt{1 - \rho^2}\sigma_2 \, dB_t^2,$$

where $W_0 = \omega$. The optimal portfolio strategy can be computed explicitly by (see Appendix)

$$\pi_t^* = \begin{cases} 1 & \text{if } \tilde{\pi} > 1 \\ \tilde{\pi} & \text{if } 0 \le \tilde{\pi} \le 1 \\ 0 & \text{if } \tilde{\pi} < 0 \end{cases}$$

where

$$\tilde{\pi} = (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)^{-1} \left(\frac{\mu_1 - \mu_2}{1 - \gamma} + \sigma_2^2 - \rho\sigma_1\sigma_2 \right).$$

When $\sigma_2 = 0$ we obtain the classical solution $\tilde{\pi} = (\mu_1 - \mu_2)/((1 - \gamma)\sigma_1^2)$ to the Merton problem with one bank account and one risky asset.

The Value-at-Risk with risk level p for a positive process X at time t is defined to be

$$\text{VaR}_t^p(X) = E[\ln X_t] - q_p(t),$$

where $q_p(t)$ is the p-quantile of $\ln X_t$. Since $W_t^{\pi^*}$ is a geometric Brownian motion we get

$$\text{VaR}_t^p(W^{\pi^*}) = -\epsilon_p \sigma^* \sqrt{t},$$

where ϵ_p is the p-quantile for a standard normal variable and

$$\sigma^* = \sqrt{\left(\rho\sigma_2 + \pi^*(\sigma_1 - \rho\sigma_2)\right)^2 + (1 - \pi^*)^2\sigma_2^2(1 - \rho^2)}.$$

The crucial factor in VaR for the optimal portfolio is its standard deviation. In most of our illustrations we shall focus on this rather than the VaR itself.

3. DEPENDENCY ON PARAMETERS IN THE OPTIMAL PORTFOLIO ALLOCATION STRATEGY

In this section we investigate the dependency of the optimal portfolio strategy on various parameters, with emphasis on the correlation and risk aversion sensitivities.

First consider the optimal allocation as a function of the correlation between the two stocks. A straightforward differentiation shows that

$$\frac{\partial \tilde{\pi}}{\partial \rho} = \sigma_1 \sigma_2 \frac{\sigma_2^2 - \sigma_1^2 + 2\frac{\mu_1 - \mu_2}{1 - \gamma}}{(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)^2}.$$

We see that $\tilde{\pi}$ is increasing with ρ when $\sigma_2^2 - \sigma_1^2 > 2(\mu_2 - \mu_1)/(1 - \gamma)$, while it is decreasing when $\sigma_2^2 - \sigma_1^2 < 2(\mu_2 - \mu_1)/(1 - \gamma)$. Note that $\tilde{\pi}$ is a constant independent of ρ when $\sigma_2^2 - \sigma_1^2 = 2(\mu_2 - \mu_1)/(1 - \gamma)$. In that case $\tilde{\pi} = 0.5$. Hence, when the volatility and the expected rate of return in the two stocks balance in a certain way we divide our wealth exactly in two equal parts, no matter how the stocks are correlated.

Next we look at $\tilde{\pi}$ as a function of the risk aversion coefficient. Computing the partial derivative of $\tilde{\pi}$ with respect to γ, we find

$$\frac{\partial \tilde{\pi}}{\partial \gamma} = \frac{\mu_1 - \mu_2}{(1 - \gamma)^2(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}.$$

From this expression it is easily seen that $\tilde{\pi}$ is an increasing of γ if $\mu_1 > \mu_2$. Moreover, the partial derivative $\frac{\partial \tilde{\pi}}{\partial \gamma}$ is increasing with γ and becomes infinite for $\gamma = 1$.

We consider some concrete examples which illustrate quantitativly the dependency on different parameters in the optimal portfolio allocation strategy. We have chosen parameters which are relevant to real market data. In the examples we focus on the variation of π_t^* as a function of the correlation, the volatility and the risk aversion.

EXAMPLE 1:
In the first example we have chosen $\mu_1 = 5\%$ and $\mu_2 = 10\%$ annually,[2] and let $\sigma_1 = 0.02$ and $\gamma = 0.5$. With these parameters $\pi_t^*(\rho)$ has been plotted in Figure 1. The different curves correspond to various values of σ_2. We notice from the figure that $\pi_t^*(\rho)$ for $\rho > 0$ changes rapidly with σ_2 when σ_2 is in the range $(0.02, 0.04)$. For values of σ_2 above this range, $\pi_t^*(\rho)$ becomes less and less sensitive to changes in σ_2 as the curves are getting denser and denser with increasing values of σ_2. In this example, π_t^* is independent of ρ and equal to 0.5 for $\sigma_2 = 0.03464$. When $\sigma_2 > 0.03464$, $\pi_t^*(\rho) > 0.5$ and increasing with ρ as long as $\pi_t^* < 1$. When $\sigma_2 < 0.03464$, $\pi_t^*(\rho) < 0.5$ and decreasing with ρ as long as $\pi_t^* > 0$. We also notice that $\pi_t^*(\rho)$ changes rapidly with ρ for $\rho > 0$ and values of σ_2 in the vicinity of $\sigma_2 = 0.03464$. Thus, when $\sigma_2 = 0.04$ we find that $\pi_t^* = 0.60$ for $\rho = 0$ and $\pi_t^* = 1$ for $\rho = 1$. Similarly, when $\sigma_2 = 0.03$ we find that $\pi_t^* = 0.38$ for $\rho = 0$ and $\pi_t^* = 0$ for $\rho = 1$. Finally, there exists a lower value for σ_2 below which π_t^* equals 0 for all values of $\rho \in [0, 1]$. In this example this lower value of σ_2 is $\sigma_2 = 0.01236$.

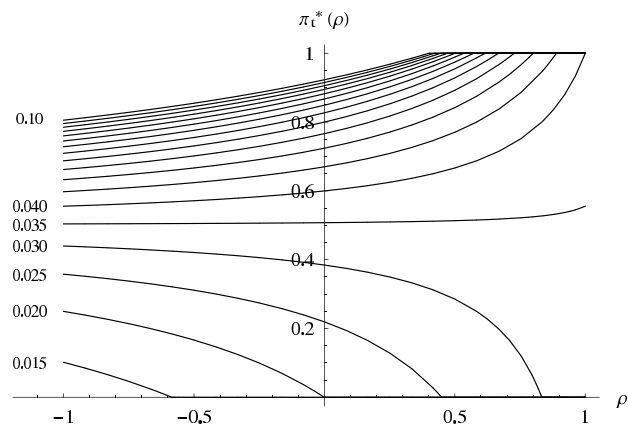[2] We have assumed 250 trading days in these examples

FIGURE 1. $\pi_t^*(\rho)$ for $\mu_1 = 5\%$ and $\mu_2 = 10\%$ annually, and fixed $\sigma_1 = 0.02$ and $\gamma = 0.5$. The family of curves corresponds to different σ_2's with the values of σ_2 indicated to the left.
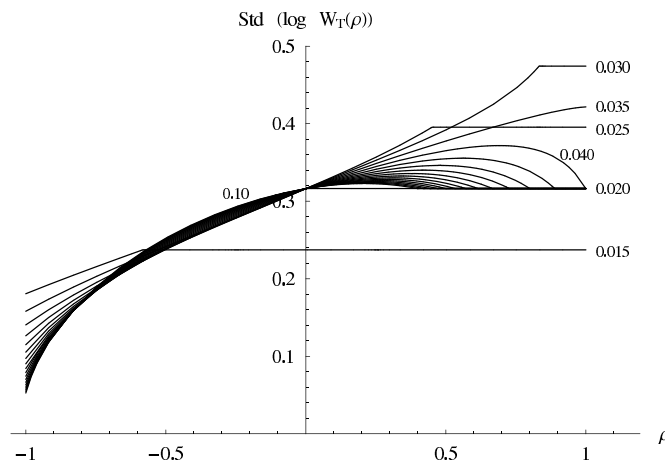

FIGURE 2. $\mathrm{Std}(\log W_T(\rho))$ for $\mu_1 = 5\%$ and $\mu_2 = 10\%$ annually, and fixed $\sigma_1 = 0.02$ and $\gamma = 0.5$. The family of curves corresponds to different σ_2's with the values of σ_2 indicated to the right.

In Figure 2 we have plotted the standard deviation of the logarithm of the wealth W_T as a function of ρ for the given values of μ_1, μ_2, σ_1, and γ. The family of curves corresponds to various values of σ_2, as indicated in the figure. We notice from the curves that the

uncertainty of the wealth process, represented by the function $Std(\log W_T(\rho))$, in general increases with the correlation ρ. However, there are a couple of special features of the plot in Figure 2 that we would like to comment on. Firstly, the curves corresponding to the values $\sigma_2 = 0.015, 0.020, 0.025$, and 0.030 are constant for certain values of ρ. This corresponds exactly to those values of ρ for which π_t^* is truncated to zero, as seen from Figure 1. Secondly, the curves corresponding to the values $\sigma_2 = 0.035, 0.040$, etc. have a maximum for some positive $\rho < 1$. Hence, for certain given values of $\mu_1, \mu_2, \sigma_1, \sigma_2$, and γ, the uncertainty of the wealth may actually decrease with increasing values of ρ. The maxima of these curves move closer and closer to $\rho = 0$ for higher and higher σ_2-values, until the curves become truncated to a constant for all positive values of ρ. This truncation of the curves corresponds to the truncation $\pi_t^* = 1$ for positive ρ's and high values of σ_2.

EXAMPLE 2:
In the second example we have taken $\mu_1 = 10\%$ and $\mu_2 = 5\%$ annually, with fixed $\sigma_1 = 0.05$ and $\gamma = 0.5$. In Figure 3 we have plotted $\pi_t^*(\rho)$ with these parameters and various values for σ_2. In this case $\pi_t^*(\rho)$ is particularly sensitive to changes in σ_2 when $\rho \geq 0.8$ and σ_2 is in the range $(0.03, 0.05)$. As in the first example, $\pi_t^*(\rho)$ becomes less and less sensitive to changes in σ_2 for larger values of σ_2. π_t^* is independent of ρ and equal to 0.5 for $\sigma_2 = 0.04123$. When $\sigma_2 > 0.04123$, $\pi_t^*(\rho) > 0.5$ and an increasing function of ρ as long as $\pi_t^* < 1$. Similarly, when $\sigma_2 < 0.04123$, $\pi_t^*(\rho) < 0.5$ and an decreasing function of ρ as long as $\pi_t^* > 0$. In this example $\pi_t^*(\rho)$ is particularly changing rapidly with ρ for $\rho > 0.8$ and values of σ_2 in the vicinity of $\sigma_2 = 0.04123$. Thus, when $\sigma_2 = 0.045$ we find that $\pi_t^* = 0.54$ for $\rho = 0$ and $\pi_t^* = 1$ for $\rho = 1$. Similarly, when $\sigma_2 = 0.04$ we find that $\pi_t^* = 0.49$ for $\rho = 0$ and $\pi_t^* = 0$ for $\rho = 1$. In the special case $\sigma_2 = 0$ we retrieve the classical solution to the Merton problem with π_t^* being a constant independent of ρ, i.e., $\pi_t^* = 0.16$. A peculiar feature of the graph is that π_t^* decreases below the value 0.16 for small values of σ_2 when ρ is large and positive. For $\rho \geq 0.8$, π_t^* dips down to zero before it increases again for even larger values of σ_2.

In Figure 4 we have plotted $Std(\log W_T(\rho))$ for the given values of μ_1, μ_2, σ_1, and γ. The different curves again correspond to various values of σ_2. As in Example 1 we notice that the function $Std(\log W_T(\rho))$ generally increases with the correlation ρ. The constant pieces of the curves corresponding to $\sigma_2 = 0.015, 0.020, 0.025, 0.030$, and 0.035 correspond to the values of ρ for which π_t^* is truncated to zero, as seen from Figure 3. We also notice that all curves with a σ_2-value less than the critical value 0.04123, for which $\pi_t^* = 0.5$, have a maximum for some positive $\rho < 1$. The maxima of these curves move closer and closer to $\rho = 1$ as σ_2 is approaching the value 0.04123. The curves corresponding to $\sigma_2 > 0.04123$ do not exhibit such a maximum for $\rho < 1$. Instead these curves are truncated to a constant above a certain value of ρ, corresponding to the truncation $\pi_t^* = 1$ for the same values of ρ.

In this example we have also made a couple of 3D-plots of the function $\pi_t^*(\rho, \gamma)$. With the given values of μ_1, μ_2 and σ_1 we have plotted $\pi_t^*(\rho, \gamma)$ for $\sigma_2 = 0.06$ in Figure 5 and for $\sigma_2 = 0.04$ in Figure 6. We notice from these figures that $\pi_t^*(\rho, \gamma)$ is an increasing function
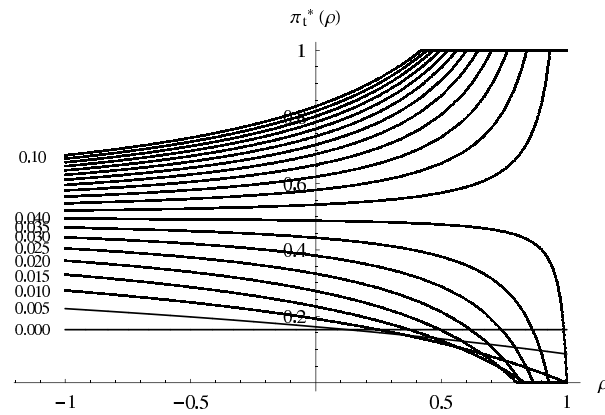
FIGURE 3. $\pi_t^*(\rho)$ for $\mu_1 = 10\%$ and $\mu_2 = 5\%$ annually, and fixed $\sigma_1 = 0.05$ and $\gamma = 0.5$. The family of curves corresponds to different σ_2's with the values of σ_2 indicated to the left.
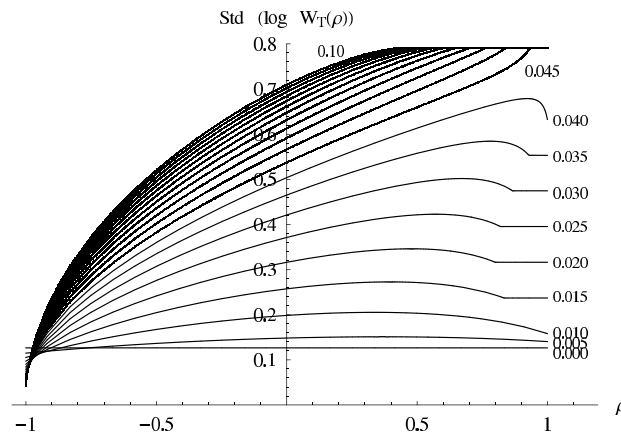

FIGURE 4. $\text{Std}(\log W_T(\rho))$ for $\mu_1 = 10\%$ and $\mu_2 = 5\%$ annually, and fixed $\sigma_1 = 0.05$ and $\gamma = 0.5$. The family of curves corresponds to different σ_2's with the values of σ_2 indicated to the right.

of γ for fixed ρ in the regions where $\pi_t^*(\rho, \gamma) < 1$. By comparing Figure 5 and Figure 6 we also notice that $\pi_t^*(\rho, \gamma)$ is very sensitive to changes in σ_2 in the range $(0.04, 0.06)$ in the region where ρ is large and positive and γ is smaller than 0.6–0.7 roughly.
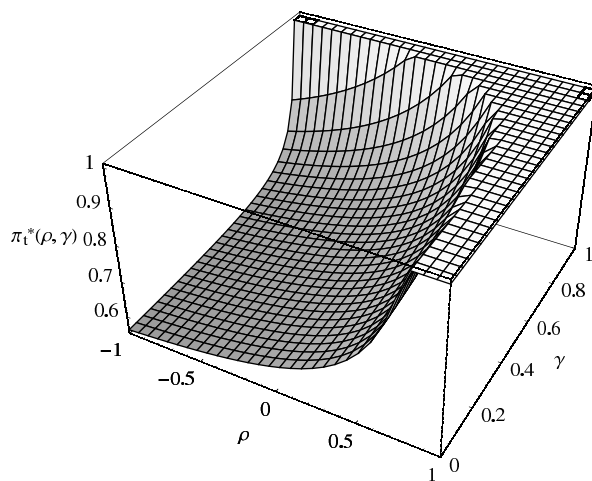
FIGURE 5. $\pi_t^*(\rho, \gamma)$ for $\mu_1 = 10\%$ and $\mu_2 = 5\%$ annually, and $\sigma_1 = 0.05$ and $\sigma_2 = 0.06$.
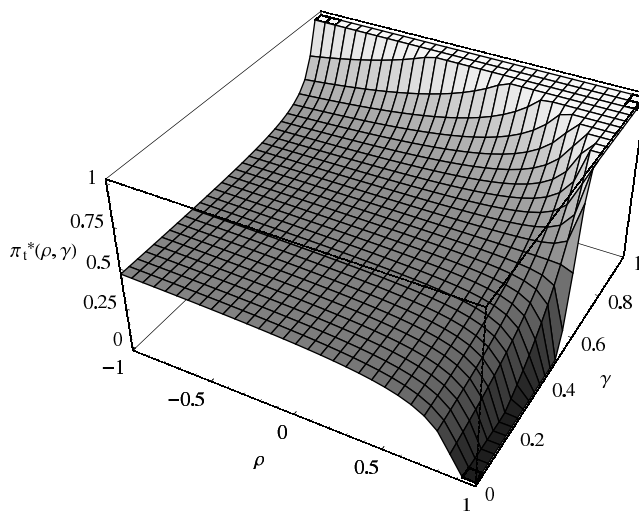

FIGURE 6. $\pi_t^*(\rho, \gamma)$ for $\mu_1 = 10\%$ and $\mu_2 = 5\%$ annually, and $\sigma_1 = 0.05$ and $\sigma_2 = 0.04$.

4. Application on a portfolio from the Oslo Stock Exchange

We consider a portfolio consiting of shares from the Agresso Gorup (AGR) and Bergesen d.y. (BEA) quoted on Oslo Stock Exchange. Over a period of 506 trading days starting January 1, 1997, we fitted the closing prices to the two dimensional geometric Brownian motion model. The parameters were estimated by maximum likelihood, giving $\mu_1 = -11.27\%$, $\sigma_1 = 0.0160$ for AGR and $\mu_2 = -12.57\%$, $\sigma_2 = 0.0101$ for BEA. The correlation was estimated to be $\rho = 0.17$.

In Figure 7 we have plotted $\pi_t^*(\gamma)$ for the assets of AGR and BEA. Also shown in Figure 7 is the graph for $\pi_t^*(\gamma)$ in the case the two assets had been totally uncorrelated ($\rho = 0$). We notice from the plot that the uncorrelated graph is higher than the correlated one for values of γ smaller than 0.325. For γ in the range $(0.325, 0.797)$ the uncorrelated graph is lower than the correlated one. For $\gamma > 0.797$ both graphs are truncated such that $\pi_t^*(\gamma) = 1$ in either case. Hence, for values of γ higher than 0.797 there is no difference between the correlated and the uncorrelated case. However, if $\gamma = 0$ we find that $\pi_t^* = 41.75\%$ in the correlated case and $\pi_t^* = 43.02\%$ in the uncorrelated case, i.e., the correlated value is 1.27% lower than the uncorrelated one. If $\gamma = 0.75$ we get an opposite result. In this case we find that $\pi_t^* = 93.23\%$ in the correlated case and $\pi_t^* = 86.59\%$ in the uncorrelated case, i.e., the correlated value is 6.64% higher than the uncorrelated one.
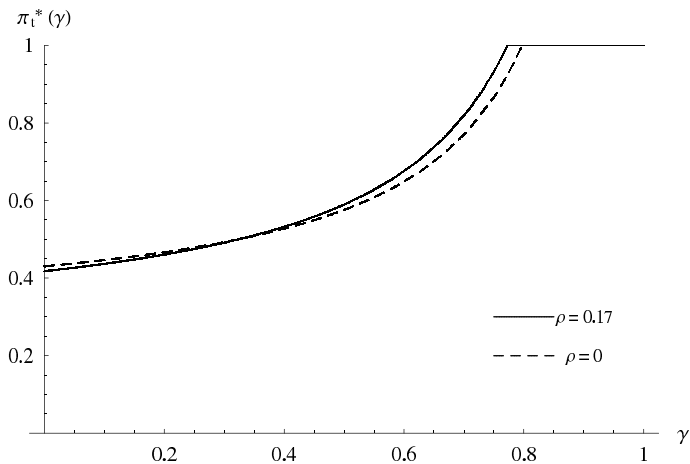

FIGURE 7. $\pi_t^*(\gamma)$ for $\mu_1 = -11.27\%$ annually and $\sigma_1 = 0.0160$ (AGR), and $\mu_2 = -12.57\%$ annually and $\sigma_2 = 0.0101$ (BEA). The correlation between AGR and BEA is $\rho = 0.17$. The dashed curve shows the values of π_t^* in the case the assets had been uncorrelated.

5. Discussion of extensions towards more realistic models

Realistic portfolio optimization models must take market frictions like liquidity and transactions costs into account. Also, statistical stylized facts of the stock prices which is not captured by the geometrical Brownian motion must be included in the modeling.

As we have seen, the Merton problem considered in this paper gives an optimal allocation strategy continuously transferring money between the two shares. This is highly unrealistic, of course, since market frictions will make such a strategy infinitely expensive. However, as Rogers [18] demonstrates, an investor performing only a finite amount of transactions will make it nearly as good as the Merton investor. Mertons problem can be generalized to treat transaction costs as well. We would like to mention among other papers Akian, Menaldi and Sulem [2] who consider multi-dimensional geometric Brownian motion market with proportional transaction costs, and Bielecki and Pliska [7] who treat both fixed and proportional fees for reallocating wealth. Note that in these papers more general functionals measuring the investor's utility are considered.

It is a well-known fact that logreturn data usually do not show a normal behaviour. This again implies that geometrical Brownian motion is not a good model for stock prices. Eberlein and Keller [11], Rydberg [18] and Prause [16] have used the generalized hyperbolic distribution in empirical studies of financial time series data. Logreturn data often have semi-heavy tails and are skew, among other stylized facts. This class of distributions seems to capture these features of logreturn data very well (see also Barndorff-Nielsen [9, 10]). Substituting the normal distribution with a generalized hyperbolic distribution leads to a stock price dynamics with Lévy process driven noise. Bank and Riedel [8], Benth, Karlsen and Reikvam [3, 4], Framstad, Øksendal and Sulem [12] and Kallsen [13] have studied the Merton problem with a risky asset driven by a Lévy process. Furthermore, in [5, 6] fixed and proportional transaction costs are included in the market model.

In Rydberg [18] and Prause [16] one finds multivariate generalized hyperbolic models for several correlated stocks. It is mathematically interesting and, in our opinion, of practical importance to find the portfolio allocation strategy for such multivariate models and study sensitivities with respect to different parameters such as the correlation and the risk aversion, etc. in order to find the critical factors. Our studies above indicate that correlation will play an important role for the allocation strategy.

Appendix

Theorem 5.1. *Let X_t^1 and X_t^2 be two assets, modelled as geometric Brownian motions,*

$$dX_t^1 = X_t^1(\mu_1\,dt + \sigma_{11}\,dB_t^1 + \sigma_{12}\,dB_t^2)$$
$$dX_t^2 = X_t^2(\mu_2\,dt + \sigma_{21}\,dB_t^1 + \sigma_{22}\,dB_t^2)$$

with parameters $\{\mu_i\}_{i=1}^2$ and $\{\sigma_{ij}\}_{i,j=1}^2$. Let $\pi_t \in [0,1]$ be the fraction of the total wealth invested in asset X_t^1, and $1 - \pi_t$ in asset X_t^2. Denote the total wealth at time t, as a

function of π_t, by W_t^π. Then

$$\pi_t^* = \begin{cases} 1 & \text{if } \tilde\pi > 1 \\ \tilde\pi & \text{if } 0 \le \tilde\pi \le 1 \\ 0 & \text{if } \tilde\pi < 0 \end{cases}$$

where

$$\tilde\pi = \frac{\mu_1 - \mu_2}{(1-\gamma)((\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2)} - \frac{\sigma_{21}(\sigma_{11} - \sigma_{21}) + \sigma_{22}(\sigma_{12} - \sigma_{22})}{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2}$$

solves the optimal control problem

$$E^{t,w}[U(W_T^{\pi^*})] = \sup_{\pi_t} E^{t,w}[U(W_T^u)]$$

where $U(x) = \frac{x^\gamma}{\gamma}, 0 < \gamma < 1$ is the HARA-utility function, $X_0^1 = x_1$, $X_0^2 = x_2$, $W_0 = w$ and $t < T$.

Proof:

The wealth process is given by

$$\begin{aligned} dW_t^\pi = {} & W_t^\pi(\mu_2 + \pi_t(\mu_1 - \mu_2)) \, dt \\ & + (\sigma_{21} + \pi_t(\sigma_{11} - \sigma_{21})) \, dB_t^1 + (\sigma_{22} + \pi_t(\sigma_{12} - \sigma_{22})) \, dB_t^2) \end{aligned}$$

and has infinitesimal generator $\mathcal{A}^\pi$ given by (on a function f)

$$\begin{aligned} \mathcal{A}^\pi f(t,x) = {} & \frac{\partial f}{\partial t} + (\mu_2 + \pi_t(\mu_1 - \mu_2))x\frac{\partial f}{\partial x} \\ & + \frac{1}{2}\{(\sigma_{21} + \pi_t(\sigma_{11} - \sigma_{21}))^2 + (\sigma_{22} + \pi_t(\sigma_{12} - \sigma_{22}))^2\}x^2\frac{\partial^2 f}{\partial x^2} \end{aligned}$$

The HJB-equation for a Markov control is

$$\sup_\pi\{\mathcal{A}^\pi\Phi(t,x)\} = 0 \quad \Phi(T,x) = U(x), \ \Phi(t,0) = U(0) \ \ t < T$$

Assume a solution of the form $\Phi(t,x) = g(t)x^\gamma$. Inserting this expression into the HJB-equation, in connection with theorem 11.2 in [15], proves the theorem. $\qquad\square$

REFERENCES

[1] RiskMetrics^{TM} - Technical document, J. P. Morgan, Reuters, Fourth Edition (1996).

[2] M. Akian, J. L. Menaldi and A. Sulem, On an investment-consumption model with transaction costs. *SIAM J. Control Optim.* **34**(1), (1996). 329-364.

[3] F. E. Benth, K. H. Karlsen and K. Reikvam, Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach. To appear in *Finance & Stochastics.*

[4] F. E. Benth, K. H. Karlsen and K. Reikvam, Merton's problem and the normal inverse Gaussian distribution - a case study. *Preprint* (2000).

[5] F. E. Benth, K. H. Karlsen and K. Reikvam, On an optimal investment-consumption model with transaction costs in a Lévy market. *Preprint* (2000).

[6] F. E. Benth, K. H. Karlsen and K. Reikvam, Intertemporal utility optimization with proportional and fixed transaction costs. Manuscript in preparation.

[7] T. R. Bielecki and S. R. Pliska, Risk sensitive asset management with transaction costs. *Finance & Stochastics* **4**(1) (2000), 1-33.

[8] P. Bank and F. Riedel, Optimal consumption choice under uncertainty with intertemporal substitution. *Prepint, Humboldt-Universität zu Berlin* (1999).

[9] O. E. Barndorff-Nielsen, Exponentially decreasing distributions for the logarithm of mparticle size, *Proc. Roy. Soc. London A* **353** (1977), 401-419.

[10] O. E. Barndorff-Nielsen, Processes of normal inverse Gaussian type, *Finance and Stochastics* **2** (1998), 41-68.

[11] E. Eberlein and U. Keller, Hyperbolic distributions in finance, *Bernoulli* **1**(3) (1995), 281-299.

[12] N. C. Framstad, B. Øksendal and A. Sulem, Optimal consumption and portfolio in a jump-diffusion market. In A. Shiryaev et al (eds.): Workshop on Mathematical Finance, INRIA Paris (1999). 281-299.

[13] J. Kallsen, Optimal portfolios for exponential Lévy processes. *Preprint, Universität Freiburg i. Br., Germany* (1999).

[14] R. Merton, Optimum consumption and portfolio rules in a continuous-time model. *J. Econom. Theory* **3**(4) (1971). 373-413.

[15] B. Øksendal, *Stochastic differential equations. Fifth edition.* 1998

[16] K. Prause, Modelling financial data using generalized hyperbolic distributions, *Preprint 48, Universität Freiburg i. Br., Germany,* (1997).

[17] L. C. G. Rogers, The relaxed investor and parameter uncertainty. *Preprint, University of Bath, UK,* (1999).

[18] T. H. Rydberg, The normal inverse Gaussian Lévy process: Simulation and approximation, *Commun. Statist.-Stochastic Models* **13**(4) (1997), 887-910.

(Fred Espen Benth)
Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
and
Norwegian Computing Center
PO Box 114 Blindern
N-0314 Oslo, Norway

E-mail address: fredb@math.uio.no

(Jon Gjerde)
Norwegian Computing Center
P.O. Box 114, Blindern
N-0314 Oslo, Norway

E-mail address: jon.gjerde@nr.no

(Sigurd Sannan)
Norwegian Computing Center
P.O. Box 114, Blindern
N-0314 Oslo, Norway

E-mail address: sigurd.sannan@nr.no